# RESTRICTED RADON TRANSFORMS AND PROJECTIONS OF PLANAR SETS

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Abstract. We establish a mixed norm estimate for the Radon transform in $\mathbb{R}^2$ when the set of directions has fractional dimension. This estimate is used to prove a result about an exceptional set of directions connected with projections of planar sets. That leads to a conjecture analogous to a well-known conjecture of Furstenberg.

## 1. Introduction

For each $\omega \in S^1$, fix $\omega^\perp$ with $\omega^\perp \perp \omega$. Define a Radon transform $R$ for functions $f$ on $\mathbb{R}^2$ by

$$Rf(t,\omega) = \int_{-1}^{1} f(t\,\omega + s\,\omega^\perp)\,ds.$$

Suppose $0 < \alpha < 1$ and fix a nonnegative Borel measure $\lambda$ on $S^1$ which is $\alpha$-dimensional in the sense that $\lambda(B(\omega,\delta)) \lesssim \delta^\alpha$ for $\omega \in S^1$. We are interested in mixed norm estimates for $R$ of the following form:

$$\Big[\int_{S^1}\Big(\int_{-1}^{1}|Rf(t,\omega)|^s dt\Big)^{q/s} d\lambda(\omega)\Big]^{1/q} \lesssim \|f\|_p. \tag{1.1}$$

Here are some conditions which are necessary for (1.1): testing on $f = \chi_{B(0,\delta)}$ shows that

$$\frac{2}{p} \le 1 + \frac{1}{s}; \tag{1.2}$$

if there is $\omega_0 \in S^1$ such that $\lambda(B(\omega_0,\delta)) \gtrsim \delta^\alpha$ for small positive $\delta$, then testing on 1 by $\delta$ rectangles centered at the origin in the direction $\omega_0^\perp$ gives

$$\frac{1}{p} \le \frac{1}{s} + \frac{\alpha}{q}; \tag{1.3}$$

if the Lebesgue measure in $S^1$ of the $\delta$-neighborhood in $S^1$ of the support of $\lambda$ is $\lesssim \delta^{1-\alpha}$, then testing on unions of 1 by $\delta$ rectangles in the directions of the support of $\lambda$ gives

$$\frac{1-\alpha}{p} \le \frac{1}{s}. \tag{1.4}$$

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Our first result is that these necessary conditions are almost sufficient:

**Theorem 1.1.** *Suppose* $p, q, r \in [1, \infty]$ *satisfy the conditions* (1.2), (1.3), *and* (1.4) *with strict inequality. Then the estimate* (1.1) *holds.*

Now suppose that $\mu$ is a nonnegative Borel measure on $\mathbb{R}^2$. If $\omega \in S^1$, define the projection $\mu_\omega$ of $\mu$ in the direction of $\omega$ by

$$\int_{\mathbb{R}} f(y)\, d\mu_\omega(y) \doteq \int_{\mathbb{R}^2} f(x \cdot \omega)\, d\mu(x),$$

where $x \cdot \omega$ denotes the inner product in $\mathbb{R}^2$. Fix $\alpha \in (0,1)$ and suppose that $\lambda$ is an $\alpha$-dimensional measure on $S^1$. Then, for $\epsilon > 0$, there is $C = C(\epsilon)$ such that

$$\int_{S^1} \frac{d\lambda(\omega)}{|\omega \cdot \omega_0|^{\alpha-\epsilon}} \leq C(\epsilon)$$

for all $\omega_0 \in S^1$. The computation

$$\int_{S^1} I_{\alpha-\epsilon}(\mu_\omega)\, d\lambda(\omega) = \int_{S^1} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{d\mu_\omega(y_1) d\mu_\omega(y_2)}{|y_1 - y_2|^{\alpha-\epsilon}} d\lambda(\omega) =$$

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{S^1} \frac{d\lambda(\omega)}{|\omega \cdot \frac{x_1 - x_2}{|x_1 - x_2|}|^{\alpha-\epsilon}} \frac{d\mu(x_1) d\mu(x_2)}{|x_1 - x_2|^{\alpha-\epsilon}} \leq C(\epsilon)\, I_{\alpha-\epsilon}(\mu)$$

is due to Kaufman [2]. Refining an earlier result of Marstrand [3], it shows that if $E \subset \mathbb{R}^2$ has dimension $\beta \leq 1$ and $p_\omega(E)$ is the projection of $E$ onto the line through the origin in the direction of $\omega$, then

$$\dim\{\omega \in S^1 : \dim p_\omega(E) < \alpha\} \leq \alpha \tag{1.5}$$

whenever $\alpha \leq \beta$. (In this note "dim" stands for Hausdorff dimension.) In particular,

$$\dim\{\omega \in S^1 : \dim p_\omega(E) < \beta\} \leq \beta. \tag{1.6}$$

The next theorem, whose analog for Minkowski dimension is trivial, complements Kaufman's results (1.5) and (1.6):

**Theorem 1.2.** *If* $\dim E = \beta \leq 1$ *then*

$$\dim\{\omega \in S^1 : \dim p_\omega(E) < \beta/2\} = 0. \tag{1.7}$$

The estimates (1.6) and (1.7) lead naturally to the conjecture that if $\alpha \leq \beta \leq 1$ then

$$\dim\{\omega \in S^1 : \dim p_\omega(E) < (\alpha + \beta)/2\} \leq \alpha. \tag{1.8}$$

One may view this conjecture as an analog of the conjecture that Furstenberg $\alpha$-sets have dimension at least $(3\alpha + 1)/2$, with (1.5) being the analog of the known $2\alpha$ lower bound for the dimension of Furstenberg sets and with (1.7) being the analog of the known $(\alpha + 1)/2$ lower bound. Indeed, (1.8) with $\beta = 1$ would imply the Furstenberg conjecture for a certain class of model Furstenberg sets. (Information about Furstenberg's conjecture is contained in [5].) The link between Theorems 1.1 and 1.2 is the fact that, formally, $\mu_\omega = R\mu(\cdot\,, \omega)$.

## 2. Proof of Theorem 1.1

The lines bounding the regions defined by (1.2) and (1.4) intersect at $(\frac{1}{p}, \frac{1}{s}) = (\frac{1}{1+\alpha}, \frac{1-\alpha}{1+\alpha})$. Then equality in (1.3) gives $\frac{1}{q} = \frac{1}{1+\alpha}$, so the important estimate is an $L^{1+\alpha} \to L^{1+\alpha}(L^{(1+\alpha)/(1-\alpha)})$ estimate. Easy estimates combined with an interpolation argument show that Theorem 1.1 will follow if we establish (1.1) for $f = \chi_E$ and a collection of triples $(p, q, r)$ which are arbitrarily close to $\big(1+\alpha, 1+\alpha, (1+\alpha)/(1-\alpha)\big)$. Standard arguments then show that it is enough to prove that if $R\chi_E(t, \omega) \geq \mu$ for

$$(t, \omega) \in F = \{(t, \omega) : \omega \in A,\, t \in B(\omega) \subset [-1, 1]\},$$

where there is some $B$ such that $B \leq m_1(B(\omega)) \leq 2B$ for $\omega \in A$, then

$$\mu^p \lambda(A)^{p/q}\, B^{p/s} \leq C(\delta)\, m_2(E)$$

if

$$p = \frac{\alpha + \delta\alpha + 1}{\delta\alpha + 1},\; q = \alpha + \delta\alpha + 1,\; s = \frac{\alpha + \delta\alpha + 1}{\delta\alpha + 1 - \alpha}$$

for small $\delta > 0$.

For each $\omega \in A$ let

$$E(\omega) = \{t\,\omega + s\,\omega^{\perp} \in E : t \in B(\omega), s \in [-1, 1]\}.$$

Since $R\chi_E(t, \omega) \geq \mu$ and $m_1(B(\omega)) \geq B$, it follows that

$$m_2(E(\omega)) \geq \mu\, B. \tag{2.1}$$

Using the change of coordinates $x \mapsto (x \cdot \omega_1, x \cdot \omega_2)$, one can check that

$$m_2\big(E(\omega_1) \cap E(\omega_2)\big) \lesssim \frac{B^2}{|\omega_1 - \omega_2|}. \tag{2.2}$$

We will bound $m_2(E)$ from below by using

$$m_2(E) \geq m_2\big(\cup_{j=1}^N E(\omega_j)\big) \geq \sum_{j=1}^N m_2(E(\omega_j)) - \sum_{1 \leq j < k \leq N} m_2\big(E(\omega_j) \cap E(\omega_k)\big) \tag{2.3}$$

for appropriately chosen $\omega_j \in A$. Fix, for the moment, a small positive number $\eta$ and consider a partitioning of $S^1$ into intervals of length about $\eta$. Since $\lambda(B(x, r)) \lesssim r^\alpha$, the $\lambda$-measure of each of these intervals is $\lesssim \eta^\alpha$. So at least, roughly, $\eta^{-\alpha}\lambda(A)$ of them must intersect $A$. Thus it is possible to choose $N \sim \eta^{-\alpha}\lambda(A)$ points $\omega_j \in A$ with $|\omega_j - \omega_k| \gtrsim \eta\,|j - k|$. Then, for any $\delta > 0$,

$$\sum_{1 \leq j < k \leq N} \frac{1}{|\omega_j - \omega_k|} \lesssim \eta^{-1} \sum_{1 \leq j < k \leq N} \frac{1}{|j - k|} \lesssim \eta^{-1} N^{1+\delta}$$

and so, by (2.2),

$$\sum_{1 \leq j < k \leq N} m_2\big(E(\omega_1) \cap E(\omega_2)\big) \leq C\, B^2 \eta^{-1} N^{1+\delta} \leq C_1 B^2 N^{1+\delta+1/\alpha} \lambda(A)^{-1/\alpha}, \tag{2.4}$$

where we have used $N \sim \eta^{-\alpha}\lambda(A)$. We would now like to choose $N$ such that

$$2\,C_1\,B^2 N^{1+\delta+1/\alpha}\lambda(A)^{-1/\alpha} \le N\,\mu\,B \le 3\,C_1\,B^2 N^{1+\delta+1/\alpha}\lambda(A)^{-1/\alpha} \tag{2.5}$$

or

$$3^{-\alpha/(1+\delta\alpha)}\Big(\frac{\mu\,B^{-1}\lambda(A)^{1/\alpha}}{C_1}\Big)^{\alpha/(\delta\alpha+1)} \le N \le 2^{-\alpha/(1+\delta\alpha)}\Big(\frac{\mu\,B^{-1}\lambda(A)^{1/\alpha}}{C_1}\Big)^{\alpha/(\delta\alpha+1)}. \tag{2.6}$$

This will be possible unless

$$\mu\,B^{-1}\lambda(A)^{1/\alpha} \lesssim 1$$

in which case

$$\mu^{\alpha/(\delta\alpha+1)}B^{-\alpha/(\delta\alpha+1)}\lambda(A)^{1/(\delta\alpha+1)} \lesssim 1$$

so that the desired inequality

$$m_2(E) \gtrsim \mu^{(\alpha+\delta\alpha+1)/(\delta\alpha+1)}\lambda(A)^{1/(\delta\alpha+1)}\,B^{(\delta\alpha+1-\alpha)/(\delta\alpha+1)} \tag{2.7}$$

follows from $m_2(E) \ge \mu\,B$ unless $F$ is empty. Now (with $N$ chosen so that (2.5) and (2.6) are valid), (2.3), (2.1), (2.4), and the left member of (2.5) give $m_2(E) \gtrsim N\,\mu\,B$. Then the left member of (2.6) gives (2.7) again.

## 3. Proof of Theorem 1.2

For $\rho > 0$, let $K_\rho$ be the kernel defined on $\mathbb{R}^d$ by $K_\rho(x) = |x|^{-\rho}\chi_{B(0,R)}(x)$ where $R = R(d)$ is positive. Suppose that the finite nonnegative Borel measure $\nu$ is a $\gamma$-dimensional measure on $\mathbb{R}^d$ in the sense that $\nu\big(B(x,\delta)\big) \le C(\nu)\,\delta^\gamma$ for all $x \in \mathbb{R}^d$ and $\delta > 0$. If $\rho < \gamma$ it follows that

$$\nu * K_\rho \in L^\infty(\mathbb{R}^d).$$

Also

$$\nu * K_\rho \in L^1(\mathbb{R}^d)$$

so long as $\rho < d$. Thus, for $\epsilon > 0$,

$$\nu * K_\rho \in L^p(\mathbb{R}^d),\ \ \rho = \gamma + \frac{1}{p}(d-\gamma) - \epsilon \tag{3.1}$$

by interpolation. The following lemma is a weak converse of this observation.

**Lemma 3.1.** *If* (3.1) *holds with* $\epsilon = 0$ *and* $p > 1$, *then* $\nu$ *is absolutely continuous with respect to Hausdorff measure of dimension* $\gamma - \epsilon$ *for any* $\epsilon > 0$. *Thus the support of* $\nu$ *has Hausdorff dimension at least* $\gamma$.

*Proof.* Recall from [1] (see p. 140) that, for $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, the norm $\|f\|^s_{p,q}$ of a distribution $f$ on $\mathbb{R}^d$ in the Besov space $B^s_{p,q}$ can be defined by

$$\|f\|^s_{pq} = \|\psi * f\|_{L^p(\mathbb{R}^d)} + \Big(\sum_{k=1}^\infty \big(2^{sk}\,\|\phi_k * f\|_{L^p(\mathbb{R}^d)}\big)^q\Big)^{1/q}$$

for certain fixed $\psi \in \mathcal{S}(\mathbb{R}^d)$, $\phi \in C_c^\infty(\mathbb{R}^d)$, and where $\phi_k(x) = 2^{kd}\phi(2^k x)$. If $\nu * K_\rho \in L^p(\mathbb{R}^d)$, then $\|\nu * \chi_{B(0,\delta)}\|_{L^p(\mathbb{R}^d)} \lesssim \delta^\rho$. It follows that $\|\nu\|^s_{pq} < \infty$ if $s < \rho - d = (\gamma - d)/p'$. Now, for $t > 0$ and $1 < p', q' < \infty$, the Besov capacity $A_{t,p',q'}(K)$ of a compact $K \subset \mathbb{R}^d$ is defined by

$$A_{t,p',q'}(K) = \inf\{\|f\|^t_{p',q'} : f \in C_c^\infty(\mathbb{R}^d),\ f \geq \chi_K\}.$$

It is shown in [4] (see p. 277) that $A_{t,p',q'}(K) \lesssim H_{d-tp'}(K)$. Thus it follows from the duality of $B^s_{p,q}$ and $B^{-s}_{p',q'}$ that

$$\nu(K) \lesssim \|\nu\|^s_{pq}\, A_{-s,p',q'}(K) \lesssim H_{d+sp'}(K) = H_{\gamma-\epsilon}(K)$$

if $s = (\gamma - d - \epsilon)/p'$.

$\square$

Now suppose that $\mu$ is a nonnegative and compactly supported Borel measure on $\mathbb{R}^2$ which is $\beta$-dimensional in the sense that $\mu\big(B(x,\delta)\big) \lesssim \delta^\beta$. If the radii $R(1)$ and $R(2)$ (in the definition of $K_\rho$) are chosen so that $R(1) = 1$ and $R(2)$ is large enough, depending on the support of $\mu$, then one can verify directly that

$$\mu_\omega * K_{(\rho-1)}(t) \lesssim \int_{-2R(2)}^{2R(2)} \mu * K_\rho\,(t\omega + s\omega^\perp)\, ds.$$

If $p, q, s$ are such that (1.1) holds and if $\rho = \beta + (2-\beta)/p - \epsilon$, so that (3.1) implies that $\mu * K_\rho \in L^p(\mathbb{R}^2)$, then a rescaling of (1.1) gives

$$\int_{S_1} \|\mu_\omega * K_{(\rho-1)}\|^q_{L^s(\mathbb{R})}\, d\lambda(\omega) < \infty. \tag{3.2}$$

If we could take $(p,q,s) = \big(1+\alpha, 1+\alpha, (1+\alpha)/(1-\alpha)\big)$ and $\epsilon = 0$ then (3.2) would yield

$$\int_{S_1} \|\mu_\omega * K_\tau\|^{1+\alpha}_{L^{(1+\alpha)/(1-\alpha)}(\mathbb{R})}\, d\lambda(\omega) < \infty$$

with $\tau = (1 - \alpha + \alpha\beta)/(1+\alpha)$. Adjusting for the fact that (3.2) actually holds only for $(p,q,s)$ close to $\big(1+\alpha, 1+\alpha, (1+\alpha)/(1-\alpha)\big)$ and with $\epsilon > 0$, it still follows that

$$\int_{S_1} \|\mu_\omega * K_\tau\|^{1+\alpha-\epsilon}_{L^{(1+\alpha-\epsilon)/(1-\alpha)}(\mathbb{R})}\, d\lambda(\omega) < \infty$$

with $\tau = (1-\alpha+\alpha\beta)/(1+\alpha) - \epsilon$ for any $\epsilon > 0$. With $\nu = \mu_\omega$, $p = (1+\alpha-\epsilon)/(1-\alpha)$, and $d = 1$, Lemma 3.1 then shows that, for any $\epsilon > 0$, the Hausdorff dimension of $\mu_\omega$'s support exceeds $\beta/2 - \epsilon$ for $\lambda$-almost all $\omega$'s. Since this is true for any $\alpha$-dimensional measure $\lambda$ and for any $\alpha \in (0,1)$, it follows that $\dim\{\omega \in S^1 : \dim p_\omega(E) < \beta/2\} = 0$ as desired.

## References


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